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ON OBJECTIVE COROTATIONAL RATES AND THEIR DEFINING SPIN TENSORS

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Abstract—In this paper, we prove a general result on objective corotational rates and their defining spin tensors: let $\mathbf{\Omega}^*$ be a spin tensor that is associated with the rotation and deformation of a deforming material body in an arbitrary manner indicated by $\mathbf{\Omega}^* = \Upsilon(\mathbf{B}, \mathbf{D}, \mathbf{W})$, where $\mathbf{B}$ and $\mathbf{D}$ and $\mathbf{W}$ are the left Cauchy–Green tensor and the stretching tensor and the vorticity tensor, respectively. Then the corotational rate of $\boldsymbol{\sigma}$ defined by the spin $\mathbf{\Omega}^*$, i.e., the tensor field $\mathring{\boldsymbol{\sigma}}^* = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\mathbf{\Omega}^* - \mathbf{\Omega}^*\boldsymbol{\sigma}$, is objective for every time-differentiable objective Eulerian symmetric tensor field $\boldsymbol{\sigma}$ if and only if the spin tensor $\mathbf{\Omega}^*$ assumes the form

$$\mathbf{\Omega}^* = \mathbf{W} + \tilde{\Upsilon}(\mathbf{B}, \mathbf{D}),$$

where $\tilde{\Upsilon}(\mathbf{B}, \mathbf{D})$ is an antisymmetric tensor-valued isotropic function. Furthermore, by virtue of certain necessary or reasonable requirements, it is found that a single antisymmetric function of two positive real variables can be introduced to characterize a general class of spin tensors defining objective corotational rates. Accordingly, a general explicit basis-free expression for the latter is established in terms of the left Cauchy–Green tensor $\mathbf{B}$, the vorticity tensor $\mathbf{W}$ and the stretching tensor $\mathbf{D}$ as well as the introduced antisymmetric function. By choosing several particular forms of the latter, it is shown that all commonly-used spin tensors are incorporated into this general expression in a natural way.

1. INTRODUCTION AND PRELIMINARIES

Objective corotational rates, which are defined by spin tensors,[1] are shown to be essential to formulation of rate-type constitutive relations, such as hypoelastic, hygrosteric and elastoplastic constitutive relations, etc. (see, e.g. Noll, 1955; Thomas, 1955a; Prager, 1960; Eringen, 1962; Truesdell and Noll, 1965; Dienes, 1979, 1986; Neale, 1981; Nemat-Nasser, 1982; Dafalias, 1983; Dubey, 1987; Metzger and Dubey, 1987; Stickforth and Wegener, 1988; Reinhardt and Dubey, 1996b; *et al.*). By now, several spin tensors and their defining objective corotational rates have been discussed in some detail (see the references just mentioned and the related literature therein). However, it seems that the general aspect of objective corotational rates and their defining spin tensors has not yet been investigated. Instead of dealing with some particular forms of spin tensors and corresponding corotational rates, as is usually done, in this article we aim to study objective corotational rates and their defining spin tensors from a general point of view. The results that will be presented are of pure kinematical character and hence independent of any particular material behaviours. To facilitate the subsequent account, in what follows we recapitulate some basic facts for kinematics of finite deformations of continua.

Consider a material body experiencing continuing finite deformation over the time interval $I \subset R$. A typical particle of this body is identified with a position vector $\mathbf{X}$ relative to a fixed reference state. The motion of the body is described by the current position vector $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ and the velocity vector $\mathbf{v} = \dot{\mathbf{x}}$. Throughout, the motion, i.e., the vector-valued

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[1] Throughout, tensor means three-dimensional second-order tensor.

function $\mathbf{x}(\mathbf{X}, t)$ is assumed to be continuously differentiable with respect to both $\mathbf{X}$ and t over the interval I.

The state of the local rotation and deformation near a neighbourhood of a particle $\mathbf{X}$ at any instant $t \in I$ is characterized by the deformation gradient

$$\mathbf{F} = \text{Grad}\,\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, \tag{1}$$

while the rate-of-change of state of the rotation and deformation near a neighbourhood of a particle $\mathbf{X}$ at any instant $t \in I$ is described by the velocity gradient

$$\mathbf{L} = \text{grad}\,\mathbf{v} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}. \tag{2}$$

For the former, the following unique left and right polar decompositions hold

$$\begin{aligned} \mathbf{F} &= \mathbf{VR} = \mathbf{RU}, \\ \mathbf{V}^2 &= \mathbf{B} = \mathbf{FF}^{\mathrm{T}}, \\ \mathbf{U}^2 &= \mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \\ \mathbf{RR}^{\mathrm{T}} &= \mathbf{I}, \end{aligned} \tag{3}$$

where the two symmetric tensors $\mathbf{V}$ and $\mathbf{B}$ and the two tensors $\mathbf{B}$ and $\mathbf{C}$ are the left and right stretch tensors and the left and right Cauchy–Green tensors, respectively, each of which is positive definite, and the proper orthogonal tensor $\mathbf{R}$ is the rotation tensor. A set of three orthonormal eigenvectors of $\mathbf{V}$ or, equivalently, $\mathbf{B}$ (resp. $\mathbf{U}$ or, equivalently, $\mathbf{C}$) are called an Eulerian triad (resp. Lagrangean triad). Throughout, $\mathbf{I}$ is used to represent the second order identity tensor. On the other hand, the following unique additive decomposition holds

$$\begin{aligned} \mathbf{L} &= \mathbf{D} + \mathbf{W}, \\ \mathbf{D} &= \tfrac{1}{2}(\mathbf{L} + \mathbf{L}^{\mathrm{T}}), \\ \mathbf{W} &= \tfrac{1}{2}(\mathbf{L} - \mathbf{L}^{\mathrm{T}}), \end{aligned} \tag{4}$$

where the tensor $\mathbf{D}$, the symmetric part of the velocity gradient $\mathbf{L}$, is the stretching tensor and the tensor $\mathbf{W}$, the antisymmetric part of the velocity gradient $\mathbf{L}$, is the vorticity tensor.

Let $\mathbf{G}$ be a time-differentiable second order Eulerian symmetric tensor field defined in the deforming body at issue. $\mathbf{G}$ is said to be objective if it obeys the following transformation rule with respect to any change of frame

$$\mathbf{x}^*(\mathbf{X}, t) = \mathbf{x}_0(t) + \mathbf{Q}(t)\mathbf{x}(\mathbf{X}, t) \Rightarrow \mathbf{G}^* = \mathbf{QGQ}^{\mathrm{T}}, \tag{5}$$

where the superscript * indicates a rotating frame characterized by the continuously time-varying rotation tensor $\mathbf{Q} = \mathbf{Q}(t)$. The latter may be in turn defined by a spin tensor $\mathbf{\Omega}^*$, i.e. a continuously differentiable time-varying antisymmetric tensor through the first order tensor differential equation

$$\dot{\mathbf{Q}}^{\mathrm{T}}\mathbf{Q} = -\mathbf{Q}^{\mathrm{T}}\dot{\mathbf{Q}} = \mathbf{\Omega}^*. \tag{6}$$

For the change of frame indicated by $(5)_1$, the following transformation formulas hold (see, e.g., Gurtin, 1981).

$$\mathbf{B}^* = \mathbf{Q}\mathbf{B}\mathbf{Q}^{\mathrm{T}}; \tag{7}$$

$$\mathbf{D}^* = \mathbf{Q}\mathbf{D}\mathbf{Q}^{\mathrm{T}}, \quad \mathbf{W}^* = \mathbf{Q}\mathbf{W}\mathbf{Q}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}. \tag{8}$$

Hence, the left Cauchy–Green tensor $\mathbf{B}$ and the stretching tensor $\mathbf{D}$ are objective, whereas the vorticity tensor $\mathbf{W}$ is not.

Let $\mathbf{\Omega}^*$ be a given spin tensor and $\boldsymbol{\sigma}$ be a time-differentiable objective Eulerian symmetric tensor field, such as the Cauchy stress field, etc. In a rotating frame defined by the spin $\mathbf{\Omega}^*$ [cf $(5)_1$ and (6)], the tensor $\boldsymbol{\sigma}$ becomes $\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}}$ and hence in the just-mentioned rotating frame, the time derivative of this tensor is given by [cf $(6)_1$]

$$\begin{aligned}\overline{(\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}})}^{\cdot} &= \mathbf{Q}\dot{\boldsymbol{\sigma}}\mathbf{Q}^{\mathrm{T}} + \dot{\mathbf{Q}}\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\boldsymbol{\sigma}\dot{\mathbf{Q}}^{\mathrm{T}} \\ &= \mathbf{Q}\mathring{\boldsymbol{\sigma}}^*\mathbf{Q}^{\mathrm{T}},\end{aligned} \tag{9}$$

where

$$\mathring{\boldsymbol{\sigma}}^* = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\mathbf{\Omega}^* - \mathbf{\Omega}^*\boldsymbol{\sigma}. \tag{10}$$

The latter is called the corotational rate defined by the spin $\mathbf{\Omega}^*$, which measures the rate of change of the tensor $\boldsymbol{\sigma}$ by an observer in a rotating frame with the spin $\mathbf{\Omega}^*$. In formulation of rate-type constitutive relations, the applied corotational rate is required to be objective so that any superimposed rigid rotating motion has no effect and therefore the principle of material objectivity is fulfilled. Towards the latter end, other kinds of objective rate measures are possible (see Oldroyd, 1950; Cotter and Rivlin, 1955; Truesdell, 1955; *et al.*). The advantage of an objective corotational rate results from its intrinsic, unique properties. Here, only two aspects are mentioned. One is that the principal invariants of stress are stationary if and only if an objective corotational rate of stress vanishes. This fact was pointed out first by Prager (1960) by virtue of the well-known case, i.e. the Zaremba–Jaumann rate (see below). It was argued by the same author that the just-mentioned fact should be incorporated in formulating elastoplastic constitutive relations. The other is, perhaps more essentially, that in a suitable rotating frame, an objective corotational rate of an objective Eulerian tensor, such as the Cauchy stress tensor, is a true time derivative. Indeed, assume that the corotational rate $\mathring{\boldsymbol{\sigma}}^*$ [cf (10)] defined by the spin tensor $\mathbf{\Omega}^*$ is objective. Then, in a rotating frame with the spin $\mathbf{\Omega}^*$ [cf $(5)_1$–(6)], the objective corotational rate $\mathring{\boldsymbol{\sigma}}^*$ defined by the spin tensor $\mathbf{\Omega}^*$ becomes $\mathbf{Q}\mathring{\boldsymbol{\sigma}}^*\mathbf{Q}^{\mathrm{T}}$ [cf $(5)_2$]. The latter is just the time derivative of the counterpart $\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}}$ of the objective Eulerian tensor $\boldsymbol{\sigma}$ in the rotating frame just mentioned [cf (9)–(10)].

Although the tensor $\boldsymbol{\sigma}$ is objective, its corotational rate $\mathring{\boldsymbol{\sigma}}^*$ need not be objective. In other words, whether or not the latter is objective depends upon the defining spin tensor $\mathbf{\Omega}^*$. The corotational rates defined by the two spin tensors $\mathbf{\Omega}^* = \mathbf{W}$ and $\mathbf{\Omega}^* = \mathbf{\Omega}^R = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}$, i.e.

$$\mathring{\boldsymbol{\sigma}}^J = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\mathbf{W} - \mathbf{W}\boldsymbol{\sigma}, \tag{11}$$

$$\mathring{\boldsymbol{\sigma}}^R = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\mathbf{\Omega}^R - \mathbf{\Omega}^R\boldsymbol{\sigma}, \tag{12}$$

provide two of several known examples of objective corotational rates, called the Zaremba–Jaumann rate (see e.g., Zaremba, 1903; Jaumann, 1911; Noll, 1955; Thomas, 1955a, 1955b; Prager, 1960; Mazur, 1961; Naghdi and Wainright, 1961; *et al.*) and the polar rate (see, e.g., Green and Naghdi, 1965; Dienes, 1979, 1987; Stickforth and Wegener, 1988; Scheidler, 1994; *et al.*).

In order to arrive at objective corotational rates $\mathring{\boldsymbol{\sigma}}^*$ through (10), it is necessary to consider such spin tensors which are associated with the rotation and deformation of the deforming material body at issue, as is shown in the above two examples. Generally, it may be assumed that the defining spin tensor $\mathbf{\Omega}^*$ in (10) depends on the rotation and deformation in a manner indicated by[2]

$$\mathbf{\Omega}^* = \mathbf{\Upsilon}(\mathbf{B}, \mathbf{D}, \mathbf{W}). \tag{13}$$

We require that the corotational rate $\mathring{\boldsymbol{\sigma}}^*$ defined by the above spin tensor through (10) be objective. Then arises a question: what is the general form of such spin tensors? It seems that until now only several particular forms of such spin tensors have been known (see, e.g. Reinhardt and Dubey, 1996a). It is the main objective of this article to investigate the general aspect of the just-mentioned question. We shall show that a definite form of spin tensors defining objective corotational rates can be derived merely from certain necessary or reasonable kinematical requirements, which is independent of any particular material behaviours. The main context of this article is arranged as follows. In Section 2, we prove that each spin tensor of the form (13) defining an objective corotational rate $\mathring{\boldsymbol{\sigma}}^*$ for any time-differentiable objective Eulerian symmetric tensor field $\boldsymbol{\sigma}$ is of the form

$$\mathbf{\Omega}^* = \mathbf{W} + \tilde{\mathbf{\Upsilon}}(\mathbf{B}, \mathbf{D}), \tag{14}$$

where the antisymmetric tensor-valued function $\tilde{\mathbf{\Upsilon}}(\mathbf{B}, \mathbf{D})$ is isotropic. In Section 3, by virtue of certain necessary or reasonable requirements, we further derive a definite form of the just-mentioned spin tensors and furthermore we show that a single antisymmetric real function of two positive real variables can be introduced to clearly and naturally characterize a general class of spin tensors, which include as particular cases all commonly-known spin tensors. Finally, in Section 4 we discuss several examples.

2. A GENERAL RESULT

The main objective of this section is to prove the following general fact.

Theorem 1. Let $\mathbf{\Omega}^*$ be a spin tensor that is associated with the rotation and deformation of a deforming body in an arbitrary manner indicated by (13). Then the corotational rate $\mathring{\boldsymbol{\sigma}}^*$ defined by this spin tensor is objective for every time-differentiable objective Eulerian symmetric tensor field defined in the same deforming body if and only if it assumes the form given by (14), where the antisymmetric tensor-valued function $\tilde{\mathbf{\Upsilon}}(\mathbf{B}, \mathbf{D})$ is isotropic. As a result, every objective corotational rate may be written in the form

$$\mathring{\boldsymbol{\sigma}}^* = \mathring{\boldsymbol{\sigma}}^J + (\boldsymbol{\sigma}\tilde{\mathbf{\Upsilon}}(\mathbf{B}, \mathbf{D}) - \tilde{\mathbf{\Upsilon}}(\mathbf{B}, \mathbf{D})\boldsymbol{\sigma}), \tag{15}$$

where $\mathring{\boldsymbol{\sigma}}^J$ is the Zaremba–Jaumann stress rate [cf (11)].

Proof. The sufficiency. Let the spin tensor $\mathbf{\Omega}^*$ be given by (14), where $\tilde{\mathbf{\Upsilon}}$ is isotropic. Then we have (15). The first term $\mathring{\boldsymbol{\sigma}}^J$ of the right-hand side of the expression (15) is just the Jaumann rate [cf (11)] and hence it is objective for any time-differentiable objective Eulerian symmetric tensor field $\boldsymbol{\sigma}$. Moreover, from the objectiveness of the tensors $\boldsymbol{\sigma}$, $\mathbf{B}$ and $\mathbf{D}$ and the isotropy of the function $\tilde{\mathbf{\Upsilon}}$ we deduce that the second term of the right-hand side of the

[2] This form implies that any two motions that are indistinguishable within a constant rigid rotation correspond to the same spin tensor $\mathbf{\Omega}^*$. Since the deformation gradient $\mathbf{F}$ and the velocity gradient $\mathbf{L}$ characterize the state of rotation and deformation and the rate-of-change of state of rotation and deformation in a deforming material body, a more general form of $\mathbf{\Omega}^*$ is given by

$$\mathbf{\Omega}^* = \hat{\mathbf{\Upsilon}}(\mathbf{F}, \mathbf{L}).$$

By means of (4) and the polar decomposition formula (3), it can be easily proved that under the just-mentioned requirement, i.e.

$$\hat{\mathbf{\Upsilon}}(\mathbf{F}\mathbf{Q}_0, \mathbf{L}) = \hat{\mathbf{\Upsilon}}(\mathbf{F}, \mathbf{L})$$

for any constant rotation tensor $\mathbf{Q}_0$ and for any $\mathbf{F}$ and $\mathbf{L}$, the above more general form is reduced to (13).

expression (15) is also objective. Thus, we conclude that the corotational rate $\mathring{\boldsymbol{\sigma}}^*$ is objective for every time-differentiable Eulerian symmetric tensor field $\boldsymbol{\sigma}$.

The necessity. Let the corotational rate $\mathring{\boldsymbol{\sigma}}^*$ be objective for every time-differentiable Eulerian symmetric tensor field $\boldsymbol{\sigma}$, where the spin tensor $\boldsymbol{\Omega}^*$ is given by (13). Denoting

$$\boldsymbol{\Omega}^* - \mathbf{W} = \boldsymbol{\Upsilon}(\mathbf{B}, \mathbf{D}, \mathbf{W}) - \mathbf{W} = \tilde{\boldsymbol{\Upsilon}}(\mathbf{B}, \mathbf{D}, \mathbf{W}),$$

we have

$$\mathring{\boldsymbol{\sigma}}^* = \mathring{\boldsymbol{\sigma}}^J + \boldsymbol{\sigma}\tilde{\boldsymbol{\Upsilon}}(\mathbf{B}, \mathbf{D}, \mathbf{W}) - \tilde{\boldsymbol{\Upsilon}}(\mathbf{B}, \mathbf{D}, \mathbf{W})\boldsymbol{\sigma}.$$

Hence

$$\mathring{\boldsymbol{\sigma}}^* - \mathring{\boldsymbol{\sigma}}^J = \boldsymbol{\sigma}\tilde{\boldsymbol{\Upsilon}}(\mathbf{B}, \mathbf{D}, \mathbf{W}) - \tilde{\boldsymbol{\Upsilon}}(\mathbf{B}, \mathbf{D}, \mathbf{W})\boldsymbol{\sigma}.$$

Since both $\mathring{\boldsymbol{\sigma}}^*$ and $\mathring{\boldsymbol{\sigma}}^J$ are objective, it is evident that $\mathring{\boldsymbol{\sigma}}^* - \mathring{\boldsymbol{\sigma}}^J$ is objective. Then, for any change of frame indicated by $(5)_1$ and (6), we infer

$$\boldsymbol{\sigma}^*\tilde{\boldsymbol{\Upsilon}}(\mathbf{B}^*, \mathbf{D}^*, \mathbf{W}^*) - \tilde{\boldsymbol{\Upsilon}}(\mathbf{B}^*, \mathbf{D}^*, \mathbf{W}^*)\boldsymbol{\sigma}^* = \mathbf{Q}(\boldsymbol{\sigma}\tilde{\boldsymbol{\Upsilon}}(\mathbf{B}, \mathbf{D}, \mathbf{W}) - \tilde{\boldsymbol{\Upsilon}}(\mathbf{B}, \mathbf{D}, \mathbf{W})\boldsymbol{\sigma})\mathbf{Q}^{\mathrm{T}}, \quad (16)$$

where $\boldsymbol{\sigma}^*$, $\mathbf{B}^*$, $\mathbf{D}^*$ and $\mathbf{W}^*$, given by $\boldsymbol{\sigma}^* = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}}$ and (7)–(8), are the respective counterparts of $\boldsymbol{\sigma}$, $\mathbf{B}$, $\mathbf{D}$ and $\mathbf{W}$ in a rotating frame with the spin $\boldsymbol{\Omega}^* = \dot{\mathbf{Q}}^{\mathrm{T}}\mathbf{Q}$. By using the identity

$$\mathbf{Q}(\mathbf{ST})\mathbf{Q}^{\mathrm{T}} = (\mathbf{QSQ}^{\mathrm{T}})(\mathbf{QTQ}^{\mathrm{T}})$$

for any orthogonal tensor $\mathbf{Q}$ and for any second order tensors $\mathbf{S}$ and $\mathbf{T}$, the eqn (16) may be recast as

$$\boldsymbol{\sigma}^*\mathbf{X} - \mathbf{X}\boldsymbol{\sigma}^* = \mathbf{O}, \quad (17)$$

where

$$\mathbf{X} = \tilde{\boldsymbol{\Upsilon}}(\mathbf{B}^*, \mathbf{D}^*, \mathbf{W}^*) - \mathbf{Q}\tilde{\boldsymbol{\Upsilon}}(\mathbf{B}, \mathbf{D}, \mathbf{W})\mathbf{Q}^{\mathrm{T}} \quad (18)$$

Since the equality (17) should hold for every possible time-differentiable objective Eulerian tensor field $\boldsymbol{\sigma}$ and for any deformation gradient $\mathbf{F}$ and any velocity gradient $\mathbf{L}$, in particular we can take $\boldsymbol{\sigma}$ as such a tensor field as $\mathbf{R}\mathbf{A}_0\mathbf{R}^{\mathrm{T}}$,[3] where the tensor $\mathbf{A}_0$ is any fixed symmetric tensor with three distinct eigenvalues. Then, the introduced tensor field $\boldsymbol{\sigma}$ always has three distinct eigenvalues at every particle of the deforming body at issue no matter what $\mathbf{B}$ and $\mathbf{L}$ are. Thus, applying the related result for the solution to the linear tensor equation $\mathbf{AX} - \mathbf{XA} = \mathbf{C}$ derived in Xiao (1995) (see also MacMillan, 1992; Guo *et al.*, 1992), we deduce $\mathbf{X} = \mathbf{O}$[4] for every particle of the deforming body, i.e.

[3] This simple form of objective Eulerian tensor field was observed and suggested to the authors by one of the reviewers of this paper.

[4] In fact, let a_i and e_i, $i = 1, 2, 3$, be the three distinct eigenvalues of $\boldsymbol{\sigma}^*$ and three corresponding independent eigenvectors of $\boldsymbol{\sigma}^*$, respectively. Then

$$\boldsymbol{\sigma}^* e_i = e_i \boldsymbol{\sigma}^* = a_i e_i.$$

Applying the latter, from (17) one can infer

$$e_i \cdot (\boldsymbol{\sigma}^*\mathbf{X} - \mathbf{X}\boldsymbol{\sigma}^*) e_j = (a_i - a_j) e_i \cdot \mathbf{X} e_j = 0$$

for any $i, j = 1, 2, 3$. Since $a_i - a_j \neq 0$ for any $i \neq j$, the latter yields $e_i \cdot \mathbf{X} e_j = 0$ for any $i \neq j$, i.e. the antisymmetric tensor $\mathbf{X}$ is the null tensor.

$$\tilde{\Upsilon}(\mathbf{B}^*, \mathbf{D}^*, \mathbf{W}^*) = \mathbf{Q}\tilde{\Upsilon}(\mathbf{B}, \mathbf{D}, \mathbf{W})\mathbf{Q}^{\mathrm{T}} \tag{19}$$

for any $\mathbf{Q}$ and for any $\mathbf{B}$ and $\mathbf{D}$ and $\mathbf{W}$.

On the other hand, for any given continuous vorticity tensor $\mathbf{W} = \mathbf{W}(t)$, there is a rotation tensor $\mathbf{S} = \mathbf{S}(t)$ such that

$$\mathbf{W} = \dot{\mathbf{S}}^{\mathrm{T}}\mathbf{S} = -\mathbf{S}^{\mathrm{T}}\dot{\mathbf{S}}, \quad \text{i.e.}\ \mathbf{W} + \mathbf{S}^{\mathrm{T}}\dot{\mathbf{S}} = \mathbf{O}. \tag{20}$$

Let $\mathbf{Q}_0$ be any constant rotation tensor independent of time. Then, by replacing $\mathbf{Q}$ in (19) and (20) with $\mathbf{Q}_0\mathbf{S}$ we derive

$$\begin{aligned} &\mathbf{Q}\tilde{\Upsilon}(\mathbf{B}, \mathbf{D}, \mathbf{W})\mathbf{Q}^{\mathrm{T}} = \tilde{\Upsilon}(\mathbf{Q}\mathbf{B}\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{D}\mathbf{Q}^{\mathrm{T}}, \mathbf{O}), \\ &\mathbf{Q} = \mathbf{Q}_0\mathbf{S}. \end{aligned} \tag{21}$$

In the above, the fact [cf (20)]

$$\begin{aligned} \mathbf{W}^* &= \mathbf{Q}(\mathbf{W} + \mathbf{Q}^{\mathrm{T}}\dot{\mathbf{Q}})\mathbf{Q}^{\mathrm{T}} \\ &= \mathbf{Q}(\mathbf{W} + \mathbf{S}^{\mathrm{T}}\mathbf{Q}_0^{\mathrm{T}}\mathbf{Q}_0\dot{\mathbf{S}})\mathbf{Q}^{\mathrm{T}} \\ &= \mathbf{Q}(\mathbf{W} + \mathbf{S}^{\mathrm{T}}\dot{\mathbf{S}})\mathbf{Q}^{\mathrm{T}} = \mathbf{O} \end{aligned}$$

has been used. Now consider any given instant t_0. Since $\mathbf{Q}_0$ is arbitrary, let

$$\mathbf{Q}_0^{\mathrm{T}} = \mathbf{S}|_{t=t_0}$$

and denote

$$\mathbf{F}_0 = \mathbf{F}|_{t=t_0}, \quad \mathbf{D}_0 = \mathbf{D}|_{t=t_0}, \quad \mathbf{W}_0 = \mathbf{W}|_{t=t_0}$$

for any given instant t_0. Then (21) produces

$$\tilde{\Upsilon}(\mathbf{B}_0, \mathbf{D}_0, \mathbf{W}_0) = \tilde{\Upsilon}(\mathbf{B}_0, \mathbf{D}_0, \mathbf{O}) \tag{22}$$

for any given instant t_0. From this we conclude that the tensor function $\tilde{\Upsilon}$ is independent of $\mathbf{W}$, i.e.

$$\tilde{\Upsilon}(\mathbf{B}, \mathbf{D}, \mathbf{W}) = \tilde{\Upsilon}(\mathbf{B}, \mathbf{D})$$

for any $\mathbf{W}$.

Furthermore, from the latter and (21) we derive

$$\tilde{\Upsilon}(\mathbf{Q}\mathbf{B}\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{D}\mathbf{Q}^{\mathrm{T}}) = \mathbf{Q}\tilde{\Upsilon}(\mathbf{B}, \mathbf{D})\mathbf{Q}^{\mathrm{T}}$$

for any rotation tensor $\mathbf{Q}$, i.e. the antisymmetric tensor-valued function $\tilde{\Upsilon}(\mathbf{B}, \mathbf{D})$ is isotropic. Q.E.D.

Theorem 1 shows that every spin tensor of the form (13) defining an objective corotational rate is a sum of the vorticity tensor $\mathbf{W}$ and an antisymmetric tensor determined by the left Cauchy–Green tensor $\mathbf{B}$ and the stretching tensor $\mathbf{D}$. This fact reveals the basic role of the vorticity tensor $\mathbf{W}$. Nevertheless, at the same time it also indicates that the latter is merely the simplest one of all possible spin tensors at issue, which corresponds to $\tilde{\Upsilon}(\mathbf{B}, \mathbf{D}) = \mathbf{O}$.

Since the concept of objective corotational rates of objective Eulerian tensor fields associated with the motion of a deforming body is purely kinematical, in Theorem 1 the corotational rate $\mathring{\boldsymbol{\sigma}}^*$ is assumed to be objective for every possible time-differentiable objective Eulerian symmetric tensor field $\boldsymbol{\sigma}$, e.g., every possible Cauchy stress field. The necessity of the representation (15) is a consequence of this assumption. On the other hand, if the tensor fields $\boldsymbol{\sigma}$ in Theorem 1 had been restricted to some particular forms of objective Eulerian tensor fields depending only on $\mathbf{B}$ and $\mathbf{L}$, e.g., $\boldsymbol{\sigma} = \mathbf{g}(\mathbf{B})$ where $\mathbf{g}$ is symmetric and isotropic, then the representation (15) would not necessarily hold without additional requirements on the tensor function $\boldsymbol{\Upsilon}$, such as the continuity and the differentiability property of $\boldsymbol{\Upsilon}$ with respect to $\mathbf{B}$ and $\mathbf{L}$, etc., since possibly for some and even all $\mathbf{B}$ and $\mathbf{L}$, the uniqueness of the tensor $\mathbf{X}$ satisfying (17) may be lost and hence $\mathbf{X} = \mathbf{O}$ may not be deduced. For this aspect, refer to the result on objective corotational strain rates given in Xiao *et al.* (1997a) for detail. An extreme case is that $\boldsymbol{\sigma}$ is restricted to the stress field of the elastic fluid, i.e. $\boldsymbol{\sigma} = -p(\rho)\mathbf{I}$. Then the term $\boldsymbol{\sigma}\boldsymbol{\Omega}^* - \boldsymbol{\Omega}^*\boldsymbol{\sigma}$ in (10) drops out for every motion of the body, and hence objectivity places no restrictions at all on the function $\boldsymbol{\Upsilon}$ in (13). Thus, no useful results are obtained, and moreover nothing can be said about any other case. In general, results derived from some particular forms of objective tensor fields $\boldsymbol{\sigma}$ apply to related particular cases only, and hence the definite and general conclusion of Theorem 1 cannot be drawn from them.

3. MATERIAL SPIN TENSORS

In this section, we derive further restrictions on the general form of the spin tensors defining objective corotational rates by virtue of the following requirements.

(i) The spin tensor $\boldsymbol{\Omega}^*$, i.e. the tensor function $\tilde{\boldsymbol{\Upsilon}}$, should be continuous with respect to $\mathbf{B}$ and moreover be continuously differentiable at $\mathbf{D} = \mathbf{O}$.

(ii) The spin tensor $\boldsymbol{\Omega}^*$ should depend linearly on any change of time scale. Specifically, let t be any given instant and let $\sigma(s)$ be a smooth increasing function with the properties: $\sigma(0) = 0$ and $\lim_{s\to\infty} \sigma(s) = \infty$. Consider two motions defined by the deformation gradients (cf Truesdell and Noll, 1965, p. 403)

$$\mathbf{F}^{(t)}(s) = \mathbf{F}(t-s), \quad \tilde{\mathbf{F}}^{(t)}(s) = \tilde{\mathbf{F}}(t-s) = \mathbf{F}(t-\sigma(s)).$$

At the instant t, the first deformation gradient $\mathbf{F}^{(t)}(s)$ corresponds to $\mathbf{D}(t)$, $\mathbf{W}(t)$ and $\boldsymbol{\Omega}^*(t)$, and the second deformation gradient $\tilde{\mathbf{F}}^{(t)}(s)$ to (cf Truesdell and Noll, 1965, p. 403)

$$\tilde{\mathbf{D}}(t) = \dot{\sigma}(0)\mathbf{D}(t), \quad \tilde{\mathbf{W}}(t) = \dot{\sigma}(0)\mathbf{W}(t) \tag{23}$$

and $\tilde{\boldsymbol{\Omega}}^*(t)$. Then, the following condition is assumed:

$$\tilde{\boldsymbol{\Omega}}^*(t) = \alpha\boldsymbol{\Omega}^*(t) \tag{24}$$

for any $\alpha = \dot{\sigma}(0) > 0$. From the latter and the continuous differentiability property of $\tilde{\boldsymbol{\Upsilon}}$ at $\mathbf{D} = \mathbf{O}$, it follows that $\tilde{\boldsymbol{\Upsilon}}$ is linear[5] in $\mathbf{D}$.

(iii) Any superimposed constant uniform dilatational deformation $\alpha\mathbf{I}$, $\alpha > 0$, has no effect on the spin $\boldsymbol{\Omega}^*$, i.e. if the deformation gradient $\mathbf{F}$ is changed to $\alpha\mathbf{F}$, then the spin tensor $\boldsymbol{\Omega}^*$ should keep unchanged, just as the velocity gradient $\mathbf{L}$ and the stretching tensor $\mathbf{D}$ and the vorticity tensor $\mathbf{W}$ do. This requirement is equivalent to

$$\tilde{\boldsymbol{\Upsilon}}(\alpha^2\mathbf{B}, \mathbf{D}) = \tilde{\boldsymbol{\Upsilon}}(\mathbf{B}, \mathbf{D}), \quad \forall\alpha > 0. \tag{25}$$

The above requirements are necessary or reasonable. First, if the first requirement is not fulfilled, then even for some C^∞ smooth motions, it is impossible to define a continuously rotating frame with the spin $\boldsymbol{\Omega}^*$, and hence the corotational rate defined by the spin tensor

[5] In fact, we have $\tilde{\boldsymbol{\Upsilon}}(\mathbf{B}, \mathbf{O}) = \mathbf{O}$ and $\tilde{\boldsymbol{\Upsilon}}(\mathbf{B}, \alpha\mathbf{D}) = \alpha\tilde{\boldsymbol{\Upsilon}}(\mathbf{B}, \mathbf{D})$, $\alpha > 0$, and hence $\tilde{\boldsymbol{\Upsilon}}(\mathbf{B}, \mathbf{D}) = \lim_{\alpha\to 0} \tilde{\boldsymbol{\Upsilon}}(\mathbf{B}, \alpha\mathbf{D})/\alpha = \mathbf{H}(\mathbf{B})[\mathbf{D}]$, the latter being linear in $\mathbf{D}$.

$\boldsymbol{\Omega}^*$ is questionable. An example is supplied by the twirl tensors of the Lagrangean and Eulerian triads. It is known (cf Scheidler, 1991) that for some C^∞ smooth motions, the latter and hence their twirl tensors are discontinuous. However, if $\boldsymbol{\Omega}^*$ is continuous with respect to $\mathbf{B}$ and $\mathbf{D}$, then for any C^2 motion, a C^1 rotating frame with the spin $\boldsymbol{\Omega}^*$ can always be defined through $(5)_1$ and (6). Next, the second requirement is equivalent to the requirement that the corotational rate defined by the spin $\boldsymbol{\Omega}^*$ should linearly depend on any change of time scale. It is evident that if this requirement is not satisfied, then the corotational rate defined by the spin $\boldsymbol{\Omega}^*$ will be inconsistent with the fact that a rate measure, such as time rate, should be linearly depend on any change of time scale. One of the consequences caused by such inconsistency is that if the corotational rate defined by $\boldsymbol{\Omega}^*$ is used to formulate hypoelastic and elastoplastic constitutive relations, then the rate-independence requirement will not be fulfilled. Finally, the third requirement arises from the consideration that any two motions that are indistinguishable within a constant uniform dilatational deformation should correspond to the same rotating frame through $(5)_1$ and (6), since any constant uniform dilatational deformation has no effect on the Lagrangean and Eulerian triads as well as the rate-of-change of deformation and rotation.

Each spin tensor $\boldsymbol{\Omega}^*$ fulfilling the above three requirements becomes a kinematical quantity of the same kind as the vorticity tensor $\mathbf{W}$, which is associated with the deformation and rotation of a deforming material body in a reasonable way. In view of this, we call each such a spin tensor a *material spin tensor* and denote it by $\boldsymbol{\Omega}^M$.

Henceforth, the notation $\Sigma_{\sigma\neq\tau}^{m}$ is used to represent the summation for $\sigma, \tau = 1, \ldots, m$ and $\sigma \neq \tau$ and this summation is assumed to be vanishing when $m = 1$. The main result of this section is as follows.

Theorem 2. Let $\chi_1, \ldots, \chi_m$ be the distinct eigenvalues of $\mathbf{B}$ and $\mathbf{B}_1, \ldots, \mathbf{B}_m$ be the corresponding subordinate eigenprojections of $\mathbf{B}$. Then each material spin tensor $\boldsymbol{\Omega}^M$ may be written in the form

$$\boldsymbol{\Omega}^M = \mathbf{W} + \sum_{\sigma\neq\tau}^{m} h\left(\frac{\chi_\sigma}{I}, \frac{\chi_\tau}{I}\right)\mathbf{B}_\sigma\mathbf{D}\mathbf{B}_\tau, I = \operatorname{tr}\mathbf{B}, \tag{26}$$

where the function $h: R^+ \times R^+ \to R: (x, y) \in R^+ \times R^+ \mapsto h(x, y) = -h(y, x)$ is a continuous antisymmetric function of two positive real variables. The latter defines the material spin tensor $\boldsymbol{\Omega}^M$ and is called the spin function.

Proof. It is evident that Theorem 2 holds when $m = 1$, since we have $\boldsymbol{\Omega}^M = \mathbf{W}$. In the following proof, we assume that $m \geqslant 2$.

Since $\tilde{\boldsymbol{\Upsilon}}$ is isotropic and linear in $\mathbf{D}$, by applying the representation theorem for skewsymmetric tensor-valued isotropic functions (see Wang, 1970; Smith, 1971; Spencer, 1971) we obtain

$$\boldsymbol{\Omega}^M = \mathbf{W} + \nu_1[\mathbf{BD}] + \nu_2[\mathbf{B}^2\mathbf{D}] + \nu_3[\mathbf{BDB}^2], \tag{27}$$

where each scalar coefficient ν_k is an isotropic invariant of $\mathbf{B}$. Here and hereafter, we denote

$$[\mathbf{ST}] = \mathbf{ST} - \mathbf{T}^T\mathbf{S}^T$$

for any two second order tensors $\mathbf{S}$ and $\mathbf{T}$.

The following facts for the eigenprojections of $\mathbf{B}$ are useful.

$$\mathbf{B}_\sigma\mathbf{B}_\tau = \begin{cases} \mathbf{O}, & \sigma \neq \tau; \\ \mathbf{B}_\sigma, & \sigma = \tau; \end{cases} \tag{28}$$

$$\sum_{\sigma=1}^{m} \mathbf{B}_\sigma = \mathbf{I}; \tag{29}$$

$$\mathbf{B} = \sum_{\sigma=1}^{m} \chi_\sigma \mathbf{B}_\sigma. \tag{30}$$

With the aid of (28)–(30), the expression (27) may be converted to

$$\begin{aligned}\mathbf{\Omega}^M - \mathbf{W} &= \sum_{\sigma,\tau=1}^{m} \mathbf{B}_\sigma(\mathbf{\Omega}^M - \mathbf{W})\mathbf{B}_\tau \\ &= \sum_{\sigma,\tau=1}^{m} \mathbf{B}_\sigma(v_1(\mathbf{BD}-\mathbf{DB}) + v_2(\mathbf{B}^2\mathbf{D}-\mathbf{DB}^2) + v_3(\mathbf{B}^2\mathbf{DB}-\mathbf{BDB}^2))\mathbf{B}_\tau \\ &= \sum_{\sigma\neq\tau}^{m} h_{\sigma\tau}\mathbf{B}_\sigma\mathbf{DB}_\tau,\end{aligned} \tag{31}$$

where

$$\begin{aligned} h_{\sigma\tau} &= (\chi_\sigma - \chi_\tau)(v_1 + (\chi_\sigma + \chi_\tau)v_2 + \chi_\sigma\chi_\tau v_3), \\ v_k &= v_k(I, II, III), \quad k = 1, 2, 3, \end{aligned} \tag{32}$$

and I, II and III are the three principal invariants of $\mathbf{B}$ [cf (40) below]. From the above facts and

$$\begin{aligned} II &= \chi_\sigma\chi_\tau + (\chi_\sigma + \chi_\tau)(I - \chi_\sigma - \chi_\tau), \\ III &= \chi_\sigma\chi_\tau(I - \chi_\sigma - \chi_\tau), \end{aligned} \tag{33}$$

for any two σ, $\tau \in \{1, \ldots, m\}$ and $\sigma \neq \tau$, where $m \geqslant 2$, we deduce that each coefficient $h_{\sigma\tau}$ can be written in

$$h_{\sigma\tau} = \tilde{h}(\chi_\sigma, \chi_\tau, I, II, III) = h(\chi_\sigma, \chi_\tau, I),$$

where the function h: $R^+ \times R^+ \times R^+ \to R$ is antisymmetric with respect to its first two variables, i.e.

$$h(x, y, z) = -h(y, x, z).$$

Hence, we have

$$\mathbf{\Omega}^M = \mathbf{W} + \sum_{\sigma\neq\tau}^{m} h(\chi_\sigma, \chi_\tau, I)\mathbf{B}_\sigma\mathbf{DB}_\tau.$$

Furthermore, the condition (25) yields

$$h(x, y, z) = h(\alpha^2 x, \alpha^2 y, \alpha^2 z) \tag{34}$$

for all $0 \neq \alpha \in R$ and all x, y, $z \in R^+$. From this it follows

$$h(x, y, z) = \tilde{h}\left(\frac{x}{z}, \frac{y}{z}\right).$$

Finally, we need to show that the continuity of the spin function h: $R^+ \times R^+ \to R$ is equivalent to the continuity of the material spin tensor $\mathbf{\Omega}^M$ defined by it. The proof can be found in Appendix 2 in Xiao *et al.* (1997a). Q.E.D.

Moreover, the following Sylvester's formula for eigenprojections holds

$$\mathbf{B}_\sigma = \delta_{1m}\mathbf{I} + \prod_{\theta \neq \sigma}^{m} \frac{\chi_\theta \mathbf{I} - \mathbf{B}}{\chi_\theta - \chi_\sigma}, \tag{35}$$

where δ_{1m} is Kronecker delta and the notation $\Pi^m_{\theta\neq\sigma}$ is used to denote the continued product for all $\theta = 1, \ldots, m$ and $\theta \neq \sigma$ when $m \geqslant 2$ and this product is assumed to be vanishing when $m = 1$. With the aid of the above formula, a general explicit basis-free expression of the material spin tensor $\mathbf{\Omega}^M$ can be established directly in terms of **W**, **B** and **D** as well as the spin function $h(x, y)$. The result is as follows.

$$\mathbf{\Omega}^M = \mathbf{W} + \mathbf{N}, \tag{36}$$

$$\mathbf{N} = \begin{cases} \mathbf{O}, & \chi_1 = \chi_2 = \chi_3, \\ \dfrac{h(\chi_1/I, \chi_2/I)}{\chi_1 - \chi_2}[\mathbf{BD}], & \chi_1 \neq \chi_2 = \chi_3, \\ v_1[\mathbf{BD}] + v_2[\mathbf{B}^2\mathbf{D}] + v_3[\mathbf{B}^2\mathbf{DB}], & \chi_1 \neq \chi_2 \neq \chi_3 \neq \chi_1 ; \end{cases} \tag{37}$$

$$\begin{aligned} v_k &= \frac{(-1)^k}{\Delta}(\chi_1^{3-k}h_{23} + \chi_2^{3-k}h_{31} + \chi_3^{3-k}h_{12}), \quad k = 1, 2, 3, \\ h_{ij} &= h(\chi_i/I, \chi_j/I), \\ \Delta &= (\chi_1 - \chi_2)(\chi_2 - \chi_3)(\chi_3 - \chi_1). \end{aligned} \tag{38}$$

The three eigenvalues of **B** (possibly repeated) are the three roots of the characteristic equation of **B**

$$\chi^3 - I\chi^2 + II\chi - III = 0$$

and therefore (cf Sawyers, 1986)

$$\begin{aligned} \chi_i &= \tfrac{1}{3}(I + 2\sqrt{I^2 - 3II}\cos\tfrac{1}{3}(\theta - 2\pi i)), \quad i = 1, 2, 3, \\ \cos\theta &= \frac{2I^3 - 9I \cdot II + 27III}{2(I^2 - 3II)^{3/2}}, \quad 0 \leqslant \theta < \pi, \end{aligned} \tag{39}$$

where

$$\begin{aligned} I &= \chi_1 + \chi_2 + \chi_3 = \mathrm{tr}\,\mathbf{B}, \\ II &= \chi_1\chi_2 + \chi_2\chi_3 + \chi_3\chi_1 = \tfrac{1}{2}(\mathrm{tr}\,\mathbf{B})^2 - \tfrac{1}{2}\mathrm{tr}\,\mathbf{B}^2, \\ III &= \det\mathbf{B} = \chi_1\chi_2\chi_3 = \tfrac{1}{6}(\mathrm{tr}\,\mathbf{B})^3 - \tfrac{1}{2}(\mathrm{tr}\,\mathbf{B})(\mathrm{tr}\,\mathbf{B})^2 + \tfrac{1}{3}\mathrm{tr}\,\mathbf{B}^3. \end{aligned} \tag{40}$$

Once the deformation gradient **F** and the velocity gradient **L** are given under any given coordinate system, the above explicit basis-free expressions enable us to directly calculate any material spin tensor $\mathbf{\Omega}^M$ without recourse to the eigenvalue/eigenvector analysis of **B**.

4. EXAMPLES

The following subclass of material spin tensors is of particular interest.

$$\mathbf{\Omega}^M = \mathbf{W} + \sum_{\sigma \neq \tau}^{m} \tilde{h}\left(\frac{\chi_\sigma}{\chi_\tau}\right) \mathbf{B}_\sigma \mathbf{D} \mathbf{B}_\tau, \tag{41}$$

where $\tilde{h}: R^+ \to R$ is a real function of a single positive real variable with the property

$$\tilde{h}(z^{-1}) = -\tilde{h}(z), \quad \forall z \in R^+. \tag{42}$$

The above subclass is already broad enough to include as particular cases all commonly-known spin tensors, as will be shown below.

Example 1. Trivially, let

$$\tilde{h}(z) = 0.$$

Then (41) yields the vorticity tensor $\mathbf{W}$, which defines the well-known Zaremba–Jaumann rate [cf (11)].

Example 2. Let

$$\tilde{h}(z) = \frac{1-\sqrt{z}}{1+\sqrt{z}}. \tag{43}$$

Then (41) yields the spin tensor defining the polar rate [cf (12)] (see Green and Naghdi, 1965; Hill, 1968, 1970, 1978; Dienes, 1979, 1986; Gurtin and Spear, 1983; Guo, 1984; Hoger and Carlson, 1984; Ogden, 1984; Hoger, 1986; Mehrabadi and Nemat-Nasser, 1987; Stickforth and Wegener, 1988; Scheidler, 1994; Reinhardt and Dubey, 1996a; *et al.*)

$$\mathbf{\Omega}^R = \dot{\mathbf{R}}\mathbf{R}^T = \mathbf{W} + \sum_{\sigma \neq \tau}^{m} \frac{\sqrt{\chi_\tau} - \sqrt{\chi_\sigma}}{\sqrt{\chi_\tau} + \sqrt{\chi_\sigma}} \mathbf{B}_\sigma \mathbf{D} \mathbf{B}_\tau. \tag{44}$$

In fact, by applying (28)–(29) and

$$\mathbf{V} = \sum_{\sigma=1}^{m} \sqrt{\chi_\sigma} \mathbf{B}_\sigma, \tag{45}$$

it can be proved that the above spin tensor satisfies the tensor equation governing the spin tensor $\mathbf{\Omega}^R$ [see, e.g. Scheidler, 1994, eqn (12)]

$$\mathbf{V}(\mathbf{\Omega}^R - \mathbf{W}) + (\mathbf{\Omega}^R - \mathbf{W})\mathbf{V} = \mathbf{D}\mathbf{V} - \mathbf{V}\mathbf{D}. \tag{46}$$

Example 3. Let

$$\tilde{h}(z) = \frac{1+z}{1-z}. \tag{47}$$

Then (41) yields the twirl tensor of the Eulerian triad [cf Hill, 1968, 1970, 1978; Gurtin and Spear, 1983; Ogden, 1984; Mehrabadi and Nemat-Nasser, 1987; Reinhardt and Dubey, 1996; *et al.*)

$$\mathbf{\Omega}^E = \mathbf{W} + \sum_{\sigma \neq \tau}^{m} \frac{\chi_\tau + \chi_\sigma}{\chi_\tau - \chi_\sigma} \mathbf{B}_\sigma \mathbf{D} \mathbf{B}_\tau. \tag{48}$$

In fact, by applying (28)–(29) it can be proved that the above spin tensor obeys the following tensor equation governing the spin tensor $\mathbf{\Omega}^E$, derived by Mehrabadi and Nemat-Nasser (1987).

$$2\mathbf{\Omega} - \mathbf{B}\mathbf{\Omega}\mathbf{B}^{-1} - \mathbf{B}^{-1}\mathbf{\Omega}\mathbf{B} = \mathbf{B}\mathbf{D}\mathbf{B}^{-1} - \mathbf{B}^{-1}\mathbf{D}\mathbf{B}, \tag{49}$$

when $\mathbf{\Omega} = \mathbf{\Omega}^E - \mathbf{W}$.

Example 4. Let

$$\tilde{h}(z) = \frac{2\sqrt{z}}{1-z}. \tag{50}$$

Then (41) yields the spin tensor $\mathbf{\Omega}^L$ and the spin tensor

$$\hat{\mathbf{\Omega}}^L = \mathbf{R}^T(\mathbf{\Omega}^L - \mathbf{W})\mathbf{R} = \sum_{\sigma \neq \tau}^{m} \frac{2\sqrt{\chi_\tau \chi_\sigma}}{\chi_\tau - \chi_\sigma} \hat{\mathbf{B}}_\sigma \mathbf{D} \hat{\mathbf{B}}_\tau, \tag{51}$$

where $\hat{\mathbf{B}}_\theta = \mathbf{R}^T\mathbf{B}_\theta\mathbf{R}$ and $\hat{\mathbf{D}} = \mathbf{R}^T\mathbf{D}\mathbf{R}$, yields the twirl tensor of the Lagrangean triad (see Hill 1968, 1970, 1978; Gurtin and Spear, 1983; Ogden, 1984; Mehrabadi and Nemat-Nasser, 1987; *et al.*), the latter being composed of three orthonormal eigenvectors of the right stretch tensor $\mathbf{U} = \mathbf{R}^T\mathbf{V}\mathbf{R}$. In fact, it can be proved that the above spin tensor satisfies the tensor equation governing the spin tensor $\hat{\mathbf{\Omega}}^L$, derived by Mehrabadi and Nemat-Nasser (1987).

$$\mathbf{U}\hat{\mathbf{\Omega}}^L + \hat{\mathbf{\Omega}}^L\mathbf{U} - \mathbf{C}\hat{\mathbf{\Omega}}^L\mathbf{U}^{-1} - \mathbf{U}^{-1}\hat{\mathbf{\Omega}}^L\mathbf{C} = 2(\mathbf{U}\hat{\mathbf{D}} - \hat{\mathbf{D}}\mathbf{U}). \tag{52}$$

Example 5. Let

$$\tilde{h}(z) = \frac{1+z}{1-z} + \frac{2}{\ln z}. \tag{53}$$

Then (41) yields the newly discovered logarithmic spin tensor (see Xiao *et al.*, 1996, 1997a, 1997b; see also Lehmann *et al.*, 1991; Reinhardt and Dubey, 1996a)

$$\mathbf{\Omega}^{\log} = \mathbf{W} + \sum_{\sigma \neq \tau}^{m} \left(\frac{\chi_\tau + \chi_\sigma}{\chi_\tau - \chi_\sigma} + \frac{2}{\ln \chi_\sigma - \ln \chi_\tau} \right) \mathbf{B}_\sigma \mathbf{D} \mathbf{B}_\tau. \tag{54}$$

The defining tensor equation of the latter is as follows (see Xiao *et al.*, 1996, 1997a, 1997b; see also Gurtin and Spear, 1983; Hoger, 1986; Lehmann *et al.*, 1991; Reinhardt and Dubey, 1996a).

$$\mathring{\mathbf{H}}^{\log} = \dot{\mathbf{H}} + \mathbf{H}\mathbf{\Omega}^{\log} - \mathbf{\Omega}^{\log}\mathbf{H} = \mathbf{D}, \tag{55}$$

where $\mathbf{H} = \ln \mathbf{V} = \frac{1}{2}\ln \mathbf{B}$ is the Hencky strain measure, i.e. the Eulerian logarithmic strain measure (see Hencky, 1928; Truesdell and Toupin, 1960; Hill 1968, 1970, 1978; Fitzjerald, 1980; Gurtin and Spear, 1983; Hoger 1986; *et al.*). The above logarithmic spin tensor possesses significant physical implications and finds applications in rate-type constitutive modelling (see Dubey and Reinhardt, 1996; Reinhardt and Dubey, 1996b; Xiao *et al.*, 1997b, 1997c).

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